# Unit M1.5

# Statically Indeterminate Systems

Readings:

CDL 2.1, 2.3, 2.4, 2.7

16.001/002 -- *"Unified Engineering"*
Department of Aeronautics and Astronautics
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# LEARNING OBJECTIVES FOR UNIT **M1.5**

*Through participation in the lectures, recitations, and work associated with Unit M1.5, it is intended that you will be able to………*

- ….**explain** the basic components of a constitutive relationship
- ….**apply** the compatibility of displacement concept for a variety of structural configurations
- ….**employ** the "Three Great Principles" to **determine** the forces and deflections of a statically indeterminate structural configuration

We have just dealt with statically determinate systems (internal or external) where the equations of equilibrium are sufficient to solve for reactions and internal forces.

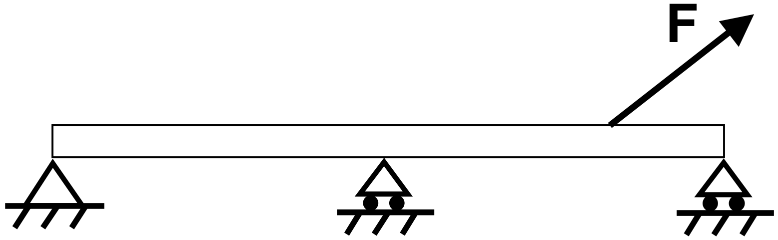
Let's go one step of complication beyond this to Statically Indeterminate Systems.

## Review of Definition

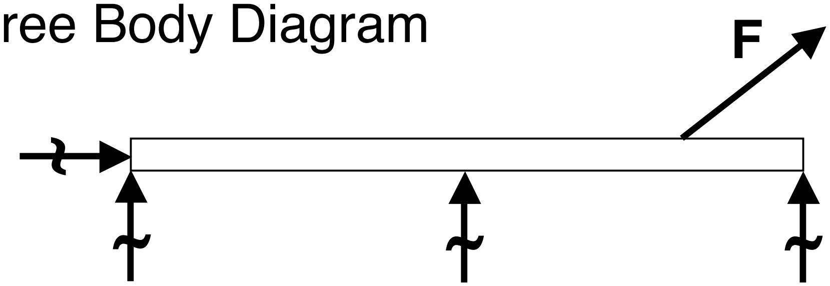
In a Statically Indeterminate System:

$$\left(\text{Number of }\begin{bmatrix}\text{rigid}\\ \text{body}\\ \text{modes}\end{bmatrix}\text{ degrees of freedom}\right) < \left(\begin{array}{l}\text{Number of}\\ \qquad\text{reactions}\end{array}\right)$$

## *Figure M1.5-1a* Example of ***Externally*** Statically Indeterminate System

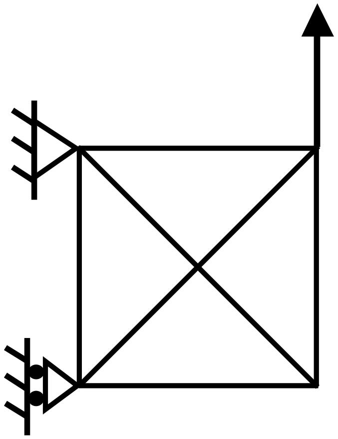


Free Body Diagram

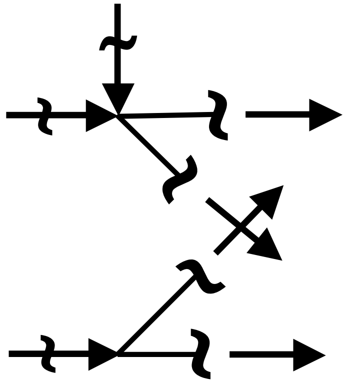


(3 dof) < (4 reactions)

## *Figure M1.5-2* ***Internally*** Statically Indeterminate



# Free Body Diagram via Method of Sections



3 equations of equilibrium (dofs)
4 internal forces ("reactions")

4 > 3 ⇒ <u>internally</u> statically indeterminate

--> Must use <u>all 3</u> "Great Principles of Solid Mechanics"

1. Equilibrium
2. Compatibility
3. Constitutive Relations

# Approach via Application of Three Great Principles

As in dealing with any structural configuration, the first step here is to draw the Free Body Diagram (this, of course, helps to show whether or not the configuration is statically indeterminate).

In the case of a Statically Determinate structure, we then just apply equilibrium

For the Statically Indeterminate case, this is more involved:

--> Approach

1. Apply equilibrium
2. Determine relations between forces and Displacements (use of constitutive relations)
3. Enforce Compatibility of Displacements

Result: Several simultaneous equations.

So:

4. Solve simultaneously for unknowns

We know what equilibrium is all about. Now, what about the other two great principles (constitutive relations, compatibility of displacements)?

Let's look at these. First consider…..

# Constitutive Relations

Definition: Constitutive relations are relations between the force applied and the resulting displacement (or vice versa)

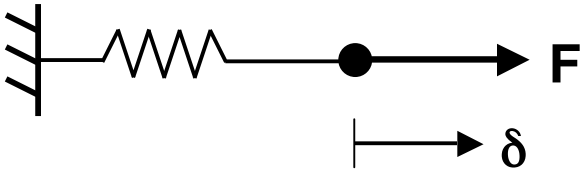
These relations depend on

- material (and certain properties)
  - → Will see this in end of Block 2 and in much of Block 3
- shape of part (i.e. geometry)

--> Consider two examples (simple)

(we will develop more and more constitutive relations throughout the year)

*Figure M1.5-4* **Example of a Spring**



F = Force
δ = displacement
k = spring constant

From physics we are familiar with the relation:

$$F = k\delta$$

Often measure k (empirically determined) to determine this constitutive relation. Can always empirically determine a constitutive relation via experiment in form:

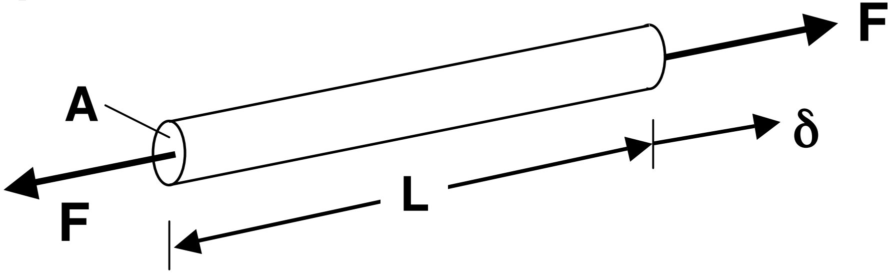
(general force) = (constitutive factor)(general displacement)

We also know from experience that a spring is stiffer (higher value of k) if

- it is thicker (geometry)
- the material is stiffer (steel spring has higher k than plastic spring)

Let's look at more of this last point by considering…..

## *Figure M1.5-5* __Example__ of a bar under axial load



Will later show that the overall change in length of the bar is:

$$\delta = \underbrace{\frac{FL}{AE}}_{\text{due to force}} + \underbrace{\alpha \Delta T L}_{\text{due to a change in temperature}}$$

where:

| | | SI |
|---|---|---|
| geometry | A = cross-section area [$L^2$] | [$m^2$] |
| | L = length of bar [L] | [m] |
| | F = applied force [$mL/T^2$] | [N] |
| material property → | E = modulus of elasticity [$mL/T^2 \cdot L^2$] | [$N/m^2$] |
| | α = coefficient of thermal expansion [L/L • temp] | [m/m°C] |
| | ΔT = temperature difference (from environment) [temp] | [°C] |

Check units:

$$\delta = \frac{\left[ML/T^2\right]\left[L\right]}{\left[L^2\right]\left[ML/(T^2 \cdot L^2)\right]} + \left[\frac{L}{L \cdot temp}\right]\left[temp\right]\left[L\right]$$

$$= \left[L\right]$$

(Note: we will derive this relationship and consider it further in Block 3)

--> Also note: This is basically where F = k$\delta$ comes from.

Set $\Delta$T = 0, then:

$$\delta = \frac{FL}{AE}$$

rearranging:

$$\boxed{\underset{\overset{\shortparallel}{\textbf{k for a solid bar!}}}{\left(\frac{AE}{L}\right)} \delta = F}$$

(in a real spring, there are additional geometric considerations)

Now let's consider the other Great Principle:

# Compatibility of Displacement

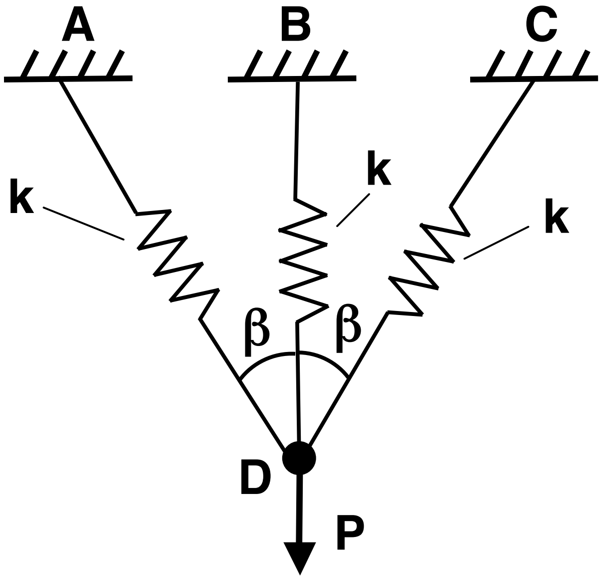
This can be stated very simply:

"configurations which are attached must have displacements consistent with the attachments"

Use geometry to determine resulting equations.

This is best defined and demonstrated by considering examples:

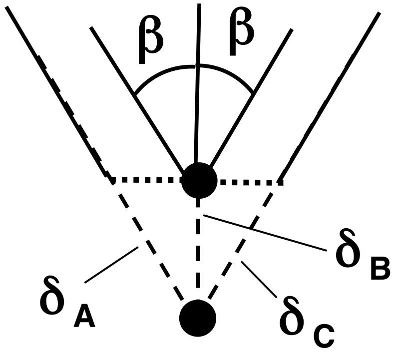
*Figure M1.5-6* **Example of three springs (or bars) attached to same point**



Compatibility here means displacement of each spring (bar) must be such that they all **conform** (are compatible) at point D where they are attached.

Consider if deformation is small such that angle β changes negligibly:

*Figure M1.5-7* **Illustration of small deformation**



Can see:

$$\delta_c \cos\beta = \delta_a \cos\beta = \delta_b$$

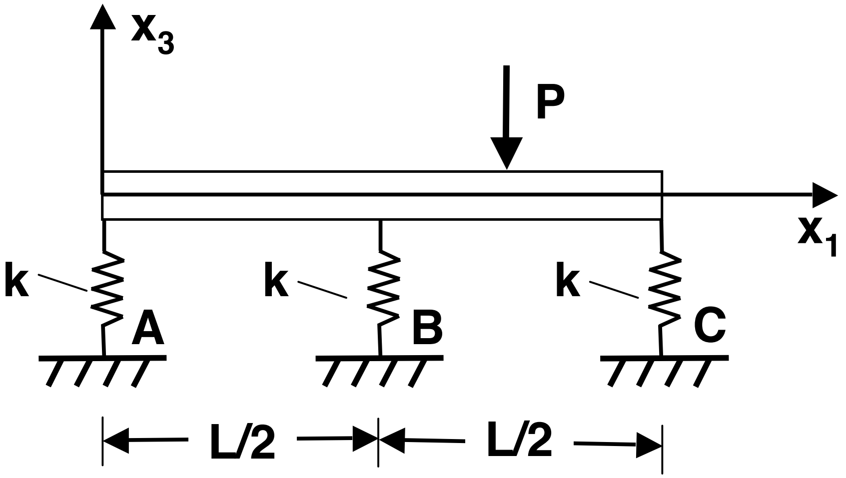
also:

$$\left|\delta_a \sin\beta\right| = \left|-\delta_c \sin\beta\right|$$

(they move vertically in opposite directions to stay attached)

What if deformation gets large?

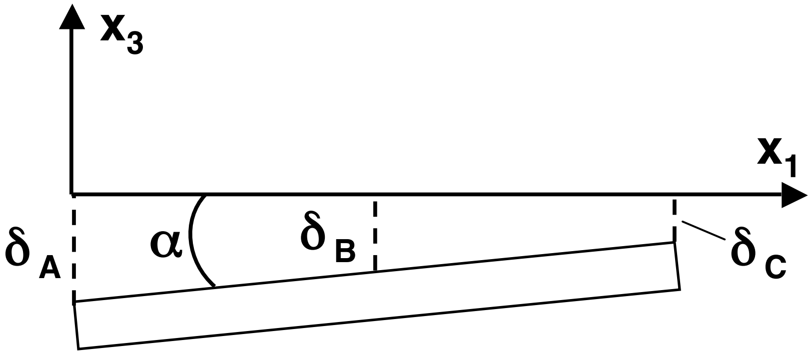
***Figure M1.5-8*** **<u>Example</u> of one bar on three springs**



Assume bar is rigid compared to springs.

Thus, bar stays straight. Any displacement in $x_3$ must thus be <u>linear</u> in $x_1$.

## *Figure M1.5-9* **Illustration of displacement compatibility for example of bar on springs**



Using geometry, define angle bar now makes with $x_1$-axis (aligned with original direction of bar)

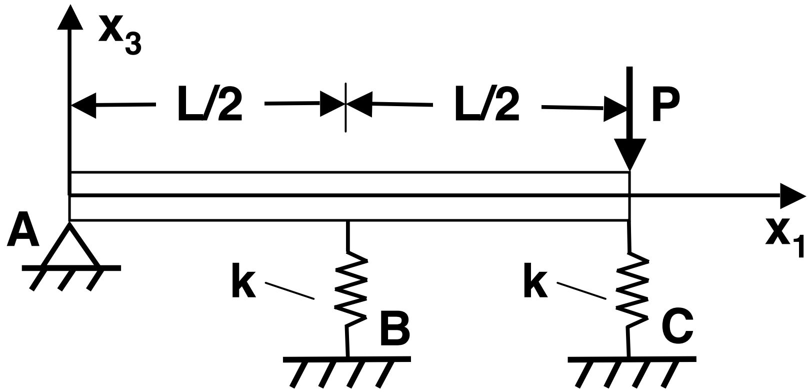
Must have:

$$\delta = a + bx_1 \qquad \text{(find a and b)}$$

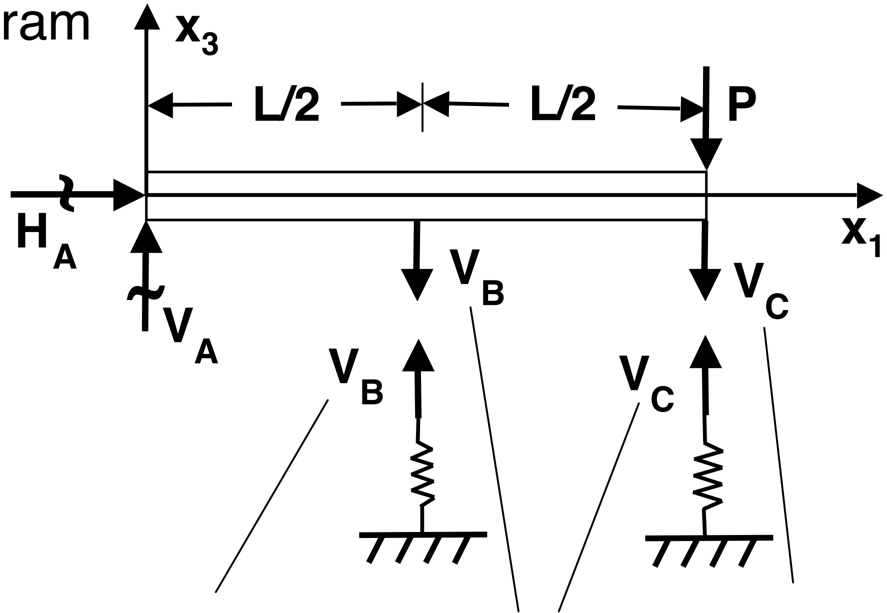
Now put things together and solve the case of

## <u>A</u> <u>Statically</u> <u>Indeterminate</u> <u>System</u>:

## *Figure M1.5-10* A rigid bar on a pivot and two springs with tip load



The first thing is
(to draw the)
Free Body Diagram



$V_B$ and $V_C$ are not reactions, but internal loads of the configuration (in the spring) defined, by our normal convention, in tension

Now apply the approach:

--> <u>Step 1</u>: Apply equilibrium

$$\sum F_H = 0 \ \xrightarrow{+} \ \Rightarrow H_A = 0 \qquad (1)$$

$$\sum F_V = 0 \ \uparrow + \ \Rightarrow V_A - V_B - V_c - P = 0 \qquad (2)$$

$$\sum M_o = 0 \ \circlearrowleft + \ \Rightarrow -V_B \ell / 2 - V_C \ell - P\ell = 0 \qquad (3)$$

gives: $$\frac{1}{2} V_b + V_C + P = 0 \qquad (3^*)$$

Equation (1) gives a solution.

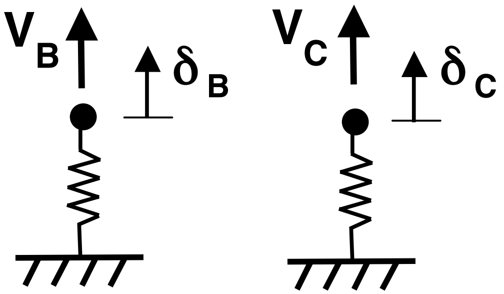
Equation (2) and (3) are 2 equations in 3 unknowns ($V_a$, $V_b$, $V_c$)

So go to:

--> <u>Step 2</u>: Determine constitutive relations

We have springs at points b and c. So:

***Figure M1.5-12*** **Illustration of loads and deflections**



$V_b = k\delta_b$ (4)

$V_c = k\delta_c$ (5)

This gives us a total of 4 equations (2, 3, 4, 5) in
5 unknowns ($V_a$, $V_b$, $V_c$, $\delta_b$, $\delta_c$).

So we go to:

--> <u>Step 3</u>: Enforce compatibility of displacements

Since the bar is rigid, the bar is straight. As we saw before, this means the displacement is linear in $x_1$:

$$\delta = m + nx_1$$

Since it is pinned at $x_1 = 0$, $\delta = 0$ there

$$\Rightarrow m = 0$$

So:

$$\delta = nx_1$$

Applying this at the two points, we get:

$$\delta_b = n \frac{\ell}{2} \qquad (6)$$

$$\delta_c = n \ell \qquad (7)$$

This gives two more equations but only one more unknown (n). So we have enough equations

Proceed to…..

--> Step 4: Solve for all the unknowns

Combining (6) and (7):

$$\delta_c = 2\,\delta_b \qquad (8)$$

Use in (4) and (5) to get:

$$V_b = k\,\delta_b \qquad (4^*)$$

$$V_c = 2\,k\,\delta_b \qquad (5^*)$$

Now put these in (2*) and (3*):

$$V_a - k\delta_b - 2\,k\delta_b - P = 0 \qquad (2^*)$$

$$\frac{1}{2}k\delta_b + 2\,k\delta_b + P = 0 \qquad (3^*)$$

progressing:

$$\frac{5}{2}\,k\delta_b = -P$$

$$\Rightarrow \delta_b = -\frac{2}{5}\,\frac{P}{k}$$

using in (2*):

$$V_a + \frac{6}{5}P - P = 0$$

$$\Rightarrow V_a = -\frac{P}{5}$$

and using in (4*), (5*):

$$V_b = -\frac{2}{5}P$$

$$V_c = -\frac{4}{5}P$$

Final result:

$$\boxed{\begin{array}{l} \left.\begin{aligned} H_a &= 0 \\ V_a &= -\frac{P}{5} \end{aligned}\right\} \text{reactions} \\ \left.\begin{aligned} \delta_b &= -\frac{2}{5}\,\frac{P}{k} \\ \delta_c &= -\frac{4}{5}\,\frac{P}{k} \end{aligned}\right\} \text{displacements} \end{array}}$$

Two closing concepts/principles that are useful in problems

1. Symmetry: If a structure is geometrically symmetric and is loaded symmetrically, then the internal forces must also be symmetric

2. Superposition: If response of materials/structures are linear (elastic materials) and undergo small (i.e., linear) deflections, then effects of different loadings can be superposed

(Thought) Example:

If know response of structure due to applied load $P$. All responses are cut in half if applied load is cut in half to $P/2$

$$P/2 + P/2 = P$$

$$\Rightarrow \text{Response } (P/2) + \text{Response } (P/2) = \text{Response } (P)$$

Similarly, say if know response of structure due to applied load $2P$, all responses are cut in half if applied load is cut in half to $P$

$$P + P = 2P$$

$$\Rightarrow \text{Response } (P) + \text{Response } (P) = \text{Response } (2P)$$

Use last equation to turn around to say if know response of structure due to applied load P, all responses are doubled if applied load is doubled

**Extend to generic case**

This concludes our Block on Statics. We will continue to use statics <u>throughout</u> Unified. In the next Block, we want to look at what goes on "inside" the structure and will thus learn about "Stress and Strain".